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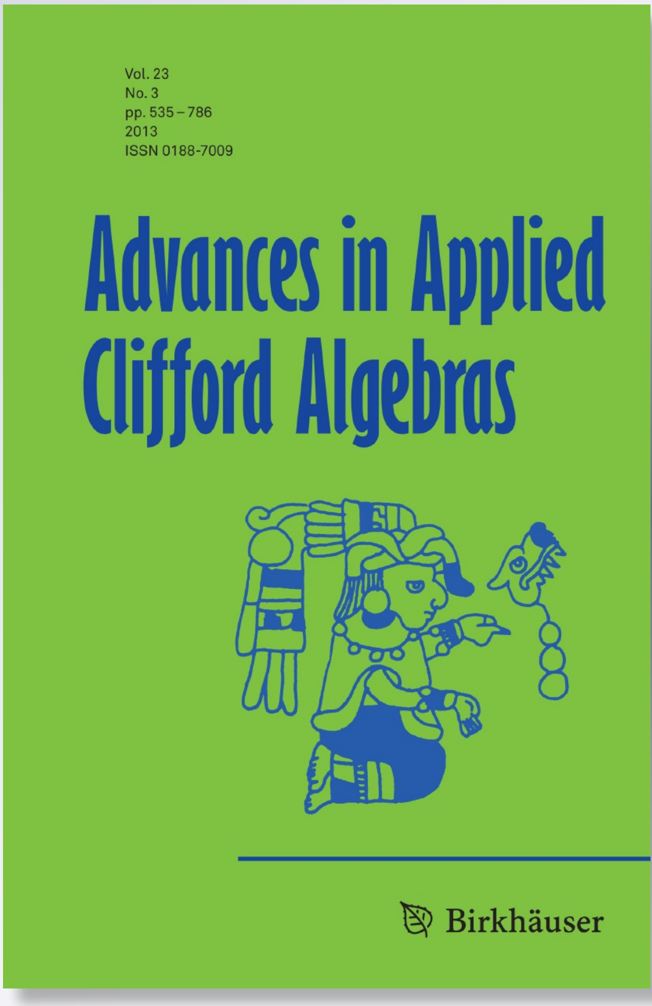
Advances in Applied Clifford Algebras

ISSN 0188-7009
Volume 23
Number 3

Adv. Appl. Clifford Algebras (2013)
23:657-671
DOI 10.1007/s00006-013-0387-3



Springer

Adv. Appl. Clifford Algebras 23 (2013), 657–671

0188-7009/030657-15
published online May 10, 2013
DOI 10.1007/s00006-013-0387-3

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Real Matrix Representations for the Complex Quaternions

Cristina Flaut* and Vitalii Shpakivskyi

Abstract. Starting from known results, due to Y. Tian in [5], referring to the real matrix representations of the real quaternions, in this paper we will investigate the left and right real matrix representations for the complex quaternions and we will give some examples in the special case of the complex Fibonacci quaternions.

Keywords. Quaternion algebra; complex Fibonacci quaternions; matrix representation.

1. Introduction

We know that each finite dimensional associative algebra A over an arbitrary field K is isomorphic with a subalgebra of the algebra $\mathcal{M}_n(K)$, with $n = \dim_K A$. Therefore, we can find a faithful representation of the algebra A in the algebra $\mathcal{M}_n(K)$. For example, the real quaternion division algebra is algebraically isomorphic to a 4×4 real matrix algebra. Starting from some results obtained by Y. Tian in [5] and in [6], in this paper we will show that the complex quaternion algebra is algebraically isomorphic to a 8×8 real matrix algebra and will investigate the properties of the obtained left and right real matrix representations for the complex quaternions. In Section 3, we will provide some examples in the special case of the complex Fibonacci quaternions.

Let K be the field $\{\begin{pmatrix} a & -b \\ b & a \end{pmatrix} \mid a, b \in \mathbb{R}\}$. The map

$$\varphi : \mathbb{C} \to K, \varphi(a + bi) = \begin{pmatrix} a & -b \\ b & a \end{pmatrix},$$

To Ioana G

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where $i^2 = -1$ is a fields morphism and $\varphi(z) = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ is called the matrix representation of the element $z = a + bi \in \mathbb{C}$.

Let $\mathbb{H}$ be the real division quaternion algebra, the algebra of the elements of the form $a = a_0 + a_1 i + a_2 j + a_3 k$, where

$$a_n \in \mathbb{R}, n \in \{0, 1, 2, 3\}, i^2 = j^2 = k^2 = -1;$$

and

$$ij = -ji = k, jk = -kj = i, ki = -ik = j.$$

$\mathbb{H}$ is an algebra over the field $\mathbb{R}$. The set $\{1, i, j, k\}$ is a basis in $\mathbb{H}$. The conjugate of the real quaternion $a = a_0 + a_1 i + a_2 j + a_3 k$ is the quaternion $\overline{a} = a_0 - a_1 i - a_2 j - a_3 k$ and $\mathbf{n}(a) = a\overline{a} = \overline{a}a$ is called *the norm* of the real quaternion a.

A complex quaternion is an element of the form $Q = c_0 + c_1 e_1 + c_2 e_2 + c_3 e_3$, where $c_n \in \mathbb{C}, n \in \{0, 1, 2, 3\}$,

$$e_n^2 = -1, \ \ n \in \{1, 2, 3\}$$

and

$$e_m e_n = -e_n e_m = \beta_{mn} e_t, \ \ \beta_{mn} \in \{-1, 1\}, m \neq n, m, n \in \{1, 2, 3\},$$

β_{mn} and e_t being uniquely determined by e_m and e_n. We denote by $\mathbb{H}_C$ the algebra of the complex quaternions, called *the complex quaternion algebra.* This algebra is an algebra over the field $\mathbb{C}$. The set $\{1, e_1, e_2, e_3\}$ is a basis in $\mathbb{H}_C$.

The map $\gamma : \mathbb{R} \to \mathbb{C}, \gamma(a) = a$ is the inclusion morphism between $\mathbb{R}$-algebras $\mathbb{R}$ and $\mathbb{C}$. We denote by $\mathbb{F}$ the $\mathbb{C}$-subalgebra of the algebra $\mathbb{H}_C$,

$$\mathbb{F} = \{Q \in \mathbb{H}_C \ \mid \ Q = c_0 + c_1 e_1 + c_2 e_2 + c_3 e_3, c_n \in \mathbb{R}, n \in \{0, 1, 2, 3\}\}.$$

By the scalar restriction, $\mathbb{F}$ became an algebra over $\mathbb{R}$, with the multiplication "$\cdot$"

$$a \cdot Q = \gamma(a) Q = aQ, a \in \mathbb{R}, Q \in \mathbb{F}.$$

We denote this algebra by $\mathbb{H}_R$. The map

$$\delta : \mathbb{H} \to \mathbb{H}_R, \delta(1) = 1, \delta(i) = e_1, \delta(j) = e_2, \delta(k) = e_3$$

and

$$\delta(a_0 + a_1 i + a_2 j + a_3 k) = a_0 + a_1 e_1 + a_2 e_2 + a_3 e_3,$$

where $a_m \in \mathbb{R}, m \in \{0, 1, 2, 3\}$ is an algebra isomorphism between the algebras $\mathbb{H}$ and $\mathbb{H}_R$.The algebra $\mathbb{H}_R$ has the same basis $\{1, e_1, e_2, e_3\}$ as the algebra $\mathbb{H}_C$. From now one, we will identify the quaternion $a_0 + a_1 i + a_2 j + a_3 k$ with the "complex" quaternion $a_0 + a_1 e_1 + a_2 e_2 + a_3 e_3$, $a_m \in \mathbb{R}, m \in \{0, 1, 2, 3\}$ and instead of $\mathbb{H}_R$ we will use $\mathbb{H}$.

It results that the element $Q \in \mathbb{H}_C, \ Q = c_0 + c_1 e_1 + c_2 e_2 + c_3 e_3, c_m \in \mathbb{C}, m \in \{0, 1, 2, 3\}$, can be written as $Q = (a_0 + ib_0) + (a_1 + ib_1)e_1 + (a_2 + ib_2)e_2 + (a_3 + ib_3)e_3$, where $a_m, b_m \in \mathbb{R}, m \in \{0, 1, 2, 3\}$ and $i^2 = -1$.

Therefore, we can write a complex quaternion under the form

$$Q = a + ib,$$

with $a, b \in \mathbb{H}$, $a = a_0 + a_1e_1 + a_2e_2 + a_3e_3, b = b_0 + b_1e_1 + b_2e_2 + b_3e_3$.

The conjugate of the complex quaternion Q is the element $\overline{Q} = c_0 - c_1e_1 - c_2e_2 - c_3e_3$. It results that

$$\overline{Q} = \overline{a} + i\overline{b}. \tag{1.1}$$

For the quaternion $a = a_0 + a_1e_1 + a_2e_2 + a_3e_3 \in \mathbb{H}$, we define the element

$$a^* = a_0 + a_1e_1 - a_2e_2 - a_3e_3. \tag{1.2}$$

We remark that

$$(a^*)^* = a \tag{1.3}$$

and

$$(a + b)^* = a^* + b^*, \tag{1.4}$$

for all $a, b \in \mathbb{H}$.

For the quaternion algebra $\mathbb{H}$, in [5], was defined the map

$$\lambda : \mathbb{H} \to \mathcal{M}_4(\mathbb{R}), \lambda(a) = \begin{pmatrix} a_0 & -a_1 & -a_2 & -a_3 \\ a_1 & a_0 & -a_3 & a_2 \\ a_2 & a_3 & a_0 & -a_1 \\ a_3 & -a_2 & a_1 & a_0 \end{pmatrix},$$

where $a = a_0 + a_1e_1 + a_2e_2 + a_3e_3 \in \mathbb{H}$ is an isomorphism between $\mathbb{H}$ and the algebra of the matrices:

$$\left\{ \begin{pmatrix} a_0 & -a_1 & -a_2 & -a_3 \\ a_1 & a_0 & -a_3 & a_2 \\ a_2 & a_3 & a_0 & -a_1 \\ a_3 & -a_2 & a_1 & a_0 \end{pmatrix}, a_0, a_1, a_2, a_3 \in \mathbb{R} \right\}.$$

We remark that the matrix $\lambda(a) \in \mathcal{M}_4(\mathbb{R})$ has as columns the coefficients in $\mathbb{R}$ of the basis $\{1, e_1, e_2, e_3\}$ for the elements $\{a, ae_1, ae_2, ae_3\}$.

The matrix $\lambda(a)$ is called *the left matrix representation* of the element $a \in \mathbb{H}$.

Analogously with the left matrix representation, for the element $a \in \mathbb{H}$ in [5], was defined *the right matrix representation*:

$$\rho : \mathbb{H} \to \mathcal{M}_4(\mathbb{R}), \ \ \rho(a) = \begin{pmatrix} a_0 & -a_1 & -a_2 & -a_3 \\ a_1 & a_0 & a_3 & -a_2 \\ a_2 & -a_3 & a_0 & a_1 \\ a_3 & a_2 & -a_1 & a_0 \end{pmatrix},$$

where $a = a_0 + a_1e_1 + a_2e_2 + a_3e_3 \in \mathbb{H}$.

We remark that the matrix $\rho(a) \in \mathcal{M}_4(\mathbb{R})$ has as columns the coefficients in $\mathbb{R}$ of the basis $\{1, e_1, e_2, e_3\}$ for the elements $\{a, e_1a, e_2a, e_3a\}$.

Proposition 1.1. [5] *For $x, y \in \mathbb{H}$ and $r \in K$ we have:*

i) $\lambda(x+y) = \lambda(x) + \lambda(y)$, $\lambda(xy) = \lambda(x)\lambda(y)$, $\lambda(rx) = r\lambda(x)$, $\lambda(1) = I_4$, $r \in K$.
ii) $\rho(x+y) = \rho(x) + \rho(y)$, $\rho(xy) = \rho(y)\rho(x)$, $\rho(rx) = r\rho(x)$, $\rho(1) = I_4$, $r \in K$.
iii) $\lambda\left(x^{-1}\right) = (\lambda(x))^{-1}$, $\rho\left(x^{-1}\right) = (\rho(x))^{-1}$, *for* $x \neq 0$.

Proposition 1.2. [5] *For $x \in \mathbb{H}$, let $\overrightarrow{x} = (a_0, a_1, a_2, a_3)^t \in \mathcal{M}_{1\times 4}(K)$, be the vector representation of the element x. Therefore for all $a, b, x \in \mathbb{H}$ the following relations are fulfilled:*

i) $\overrightarrow{ax} = \lambda(a)\overrightarrow{x}$.
ii) $\overrightarrow{xb} = \rho(b)\overrightarrow{x}$.
iii) $\overrightarrow{axb} = \lambda(a)\rho(b)\overrightarrow{x} = \rho(b)\lambda(a)\overrightarrow{x}$.
iv) $\rho(b)\lambda(a) = \lambda(a)\rho(b)$.
v) $\det(\lambda(x)) = \det(\rho(x)) = (n(x))^2$.

For details about the matrix representations of the real quaternions, the reader is referred to [5].

2. Main Results

Let θ be the matrix $\theta = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} = \lambda(e_1) = \lambda(i)$. The matrix

$$\Gamma(Q) = \begin{pmatrix} \lambda(a) & -\lambda(b^*) \\ \lambda(b) & \lambda(a^*) \end{pmatrix},$$

where $Q = a + ib$ is a complex quaternion, with $a = a_0 + a_1e_1 + a_2e_2 + a_3e_3 \in \mathbb{H}, b = b_0 + b_1e_1 + b_2e_2 + b_3e_3 \in \mathbb{H}$ and $i^2 = -1$, is called *the left real matrix representation for the complex quaternion* Q. The *right real matrix representation* for the complex quaternion Q is the matrix:

$$\Theta(Q) = \begin{pmatrix} \rho(a) & -\rho(b) \\ \rho(b^*) & \rho(a^*) \end{pmatrix}.$$

We remark that $\Gamma(Q), \Theta(Q) \in \mathcal{M}_8(\mathbb{R})$.

Now, let M be the matrix

$$M = (1, -e_1, -e_2, -e_3)^t.$$

Proposition 2.1. *If $a = a_0 + a_1e_1 + a_2e_2 + a_3e_3 \in \mathbb{H}$, we have:*

i) $\lambda(a)M = Ma$.
ii) $\theta M = Me_1$.
iii) $\lambda(ia) = \theta\lambda(a)$ *and* $\lambda(ai) = \lambda(a)\theta$.

Proof. i) $\lambda(a)M = \begin{pmatrix} a_0 & -a_1 & -a_2 & -a_3 \\ a_1 & a_0 & -a_3 & a_2 \\ a_2 & a_3 & a_0 & -a_1 \\ a_3 & -a_2 & a_1 & a_0 \end{pmatrix} \begin{pmatrix} 1 \\ -e_1 \\ -e_2 \\ -e_3 \end{pmatrix}$

$= \begin{pmatrix} a_0+a_1e_1+a_2e_2+a_3e_3 \\ a_1\text{-}a_0e_1+a_3e_2\text{-}a_2e_3 \\ a_2\text{-}a_3e_1\text{-}a_0e_2+a_1e_3 \\ a_3+a_2e_1\text{-}a_1e_2\text{-}a_0e_3 \end{pmatrix} = \begin{pmatrix} a_0+a_1e_1+a_2e_2+a_3e_3 \\ \text{-}e_1(a_0+a_1e_1+a_2e_2+a_3e_3) \\ \text{-}e_2(a_0+a_1e_1+a_2e_2+a_3e_3) \\ \text{-}e_3(a_0+a_1e_1+a_2e_2+a_3e_3) \end{pmatrix}$

$= \begin{pmatrix} 1 \\ -e_1 \\ -e_2 \\ -e_3 \end{pmatrix} a = Ma.$

ii) $\theta M = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -e_1 \\ -e_2 \\ -e_3 \end{pmatrix} = \begin{pmatrix} e_1 \\ 1 \\ e_3 \\ -e_2 \end{pmatrix}$

$= \begin{pmatrix} 1 \\ -e_1 \\ -e_2 \\ -e_3 \end{pmatrix} e_1 = Me_1.$

iii) For $a = a_0 + a_1e_1 + a_2e_2 + a_3e_3 \in \mathbb{H}$, we have $ia = -a_1 + a_0e_1 - a_3e_2 + a_2e_3$. It results that

$$\lambda(ia) = \begin{pmatrix} -a_1 & -a_0 & a_3 & -a_2 \\ a_0 & -a_1 & -a_2 & -a_3 \\ -a_3 & a_2 & -a_1 & -a_0 \\ a_2 & a_3 & a_0 & -a_1 \end{pmatrix}.$$

Since $\theta\lambda(a) = \begin{pmatrix} 0 & \text{-}1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \text{-}1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} a_0 & -a_1 & -a_2 & -a_3 \\ a_1 & a_0 & -a_3 & a_2 \\ a_2 & a_3 & a_0 & -a_1 \\ a_3 & -a_2 & a_1 & a_0 \end{pmatrix}$

$= \begin{pmatrix} -a_1 & -a_0 & a_3 & -a_2 \\ a_0 & -a_1 & -a_2 & -a_3 \\ -a_3 & a_2 & -a_1 & -a_0 \\ a_2 & a_3 & a_0 & -a_1 \end{pmatrix}$, we obtain the asked relation. □

Proposition 2.2. *Let $a, x \in \mathbb{H}$ be two quaternions, then the following relations are true:*

i) $a^*i = ia$, *where* $i^2 = -1$.
ii) $ai = ia^*$, *where* $i^2 = -1$.
iii) $-a^* = iai$, *where* $i^2 = -1$.
iv) $(xa)^* = x^*a^*$.
v) *For* $X, A \in \mathbb{H}_C, X = x + iy, A = a + ib$, *we have*

$$XA = xa - y^*b + i(x^*b + ya).$$

Proof. Relations from i), ii), iii) are obviously.

iv) From ii), it results $(xa)^* = -i\,(xa)\,i = -ixai = (ixi)(iai) = x^*a^*$.

v) We obtain

$$XA = (x+iy)\,(a+ib) = xa+xib+iya+iyib = xa-y^*b+i\,(x^*b+ya)\,. \qquad \square$$

Proposition 2.3. *For $X, A \in \mathbb{H}_C$, $X = x+iy$, $A = a+ib$, we have* $\Gamma(XA) = \Gamma(X)\,\Gamma(A)\,.$

Proof. From Proposition 1.2 i) and Proposition 2.2 iv), it results that

$$\Gamma(X)\,\Gamma(A) = \begin{pmatrix} \lambda(x) & -\lambda(y^*) \\ \lambda(y) & \lambda(x^*) \end{pmatrix} \begin{pmatrix} \lambda(a) & -\lambda(b^*) \\ \lambda(b) & \lambda(a^*) \end{pmatrix}$$

$$= \begin{pmatrix} \lambda(x)\,\lambda(a) - \lambda(y^*)\,\lambda(b) & -\lambda(x)\,\lambda(b^*) - \lambda(y^*)\,\lambda(a^*) \\ \lambda(y)\,\lambda(a) + \lambda(x^*)\,\lambda(b) & -\lambda(y)\,\lambda(b^*) + \lambda(x^*)\,\lambda(a^*) \end{pmatrix}$$

$$= \begin{pmatrix} \lambda(xa - y^*b) & -\lambda(xb^* + y^*a^*) \\ \lambda(ya + x^*b) & \lambda(-yb^* + x^*a^*) \end{pmatrix}.$$

$$\Gamma(XA) = \begin{pmatrix} \lambda(xa - y^*b) & -\lambda((x^*b + ya)^*) \\ \lambda(x^*b + ya) & \lambda((xa - y^*b)^*) \end{pmatrix}$$

$$= \begin{pmatrix} \lambda(xa - y^*b) & -\lambda(xb^* + y^*a^*) \\ \lambda(ya + x^*b) & \lambda(x^*a^* - yb^*) \end{pmatrix}. \qquad \square$$

Definition 2.4. For $X \in \mathbb{H}_C, X = x+iy$, we denote by

$$\overrightarrow{X} = (\overrightarrow{x}, \overrightarrow{y})^t \in \mathcal{M}_{8\times 1}(\mathbb{R})$$

the vector representation of the element X, where $x = x_0 + x_1e_1 + x_2e_2 + x_3e_3 \in \mathbb{H}$, $y = y_0 + y_1e_1 + y_2e_2 + y_3e_3 \in \mathbb{H}$ and $\overrightarrow{x} = (x_0, x_1, x_2, x_3)^t \in \mathcal{M}_{4\times 1}(\mathbb{R})$, $\overrightarrow{y} = (y_0, y_1, y_2, y_3)^t \in \mathcal{M}_{4\times 1}(\mathbb{R})$ are the vector representations for the quaternions x and y, as was defined in Proposition 1.2.

Proposition 2.5. *Let $X \in \mathbb{H}_C$, $X = x+iy$, $x, y \in \mathbb{H}$, then:*

i) $\overrightarrow{X} = \Gamma(X)\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, *where $1 = I_4 \in \mathcal{M}_4(\mathbb{R})$ is the identity matrix and $0 = O_4 \in \mathcal{M}_4(\mathbb{R})$ is the zero matrix.*

ii) $\overrightarrow{AX} = \Gamma(A)\,\overrightarrow{X}$.

iii) $\alpha\overrightarrow{y^*} = \overrightarrow{y}$, *where* $\alpha = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \in \mathcal{M}_4(\mathbb{R})\,.$

iv) $\alpha^2 = I_4$.

Proof. i) $\Gamma(X)\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \lambda(x) & -\lambda(y^*) \\ \lambda(y) & \lambda(x^*) \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \lambda(x) \\ \lambda(y) \end{pmatrix}$

$$= \begin{pmatrix} \lambda(1\cdot x) \\ \lambda(1\cdot y) \end{pmatrix} = \begin{pmatrix} \lambda(1)\,\overrightarrow{x} \\ \lambda(1)\,\overrightarrow{y} \end{pmatrix} = \begin{pmatrix} \overrightarrow{x} \\ \overrightarrow{y} \end{pmatrix}.$$

ii) From i), we obtain that

$$\overrightarrow{AX} = \Gamma(AX)\begin{pmatrix}1\\0\end{pmatrix} = \Gamma(A)\Gamma(X)\begin{pmatrix}1\\0\end{pmatrix} = \Gamma(A)\overrightarrow{X}.$$

iii) $$\alpha\overrightarrow{y^*} = \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix}\begin{pmatrix}y_0\\y_1\\-y_2\\-y_3\end{pmatrix} = \begin{pmatrix}y_0\\y_1\\y_2\\y_3\end{pmatrix} = \overrightarrow{y}.$$ □

Proposition 2.6. *Let* M_8 *be the matrix* $M_8 = \begin{pmatrix}\theta M\\-M\end{pmatrix}$, *then* $-\frac{1}{4}M_8^t M_8 = 1$.

Proof. It results

$$M_8^t M_8 = \begin{pmatrix} e_1 & -1 & e_3 & e_2 & -1 & e_1 & e_2 & e_3\end{pmatrix}\begin{pmatrix}e_1\\-1\\e_3\\e_2\\-1\\e_1\\e_2\\e_3\end{pmatrix} = -4.$$ □

Theorem 2.7. *Let* $Q \in \mathbb{H}_C$ *be a complex quaternion. With the above notations, the following relations are fulfilled:*

i) $\Gamma^t(Q^*)M_8 = M_8 Q$, *where* $Q = x + iy$, $Q^* = x^* + iy$, $x, y \in \mathbb{H}$.
ii) $Q = -\frac{1}{4}M_8^t\Gamma(Q^*)M_8$.

Proof. i) Let Q be a complex quaternion. From Proposition 2.1 i) and ii), we obtain:

$$\Gamma^t(Q^*)M_8 = \begin{pmatrix}\lambda(x^*) & \lambda(y)\\ -\lambda(y^*) & \lambda(x)\end{pmatrix}\begin{pmatrix}\theta M\\-M\end{pmatrix}$$
$$= \begin{pmatrix}\lambda(x^*)\theta M - \lambda(y)M\\ -\lambda(y^*)\theta M - \lambda(x)M\end{pmatrix} = \begin{pmatrix}\lambda(x^* i - y)M\\ -\lambda(y^* i + x)M\end{pmatrix}$$
$$= \begin{pmatrix}\lambda(ix + iiy)M\\ -\lambda(iy + x)M\end{pmatrix} = \begin{pmatrix}\lambda(i(x+iy))M\\ -M(x+iy)\end{pmatrix}$$
$$= \begin{pmatrix}\theta\lambda(x+iy)M\\ -M(x+iy)\end{pmatrix} = \begin{pmatrix}\theta M(x+iy)\\ -M(x+iy)\end{pmatrix}\begin{pmatrix}\theta M\\ -M\end{pmatrix}(x+iy) = M_8 Q.$$

ii) If we multiply the relation $\Gamma^t(Q^*)M_8 = M_8 Q$ to the left side with $-\frac{1}{4}M_8^t$, we obtain $Q = -\frac{1}{4}M_8^t\Gamma^t(Q^*)M_8$. □

Proposition 2.8. *For* $X, A \in \mathbb{H}_C$, $X = x + iy$, $A = a + ib$, *we have*

$$\Theta(XA) = \Theta(A)\Theta(X).$$

Proof. Using Proposition 1.1 ii), Proposition 2.2 iv), relations 1.3 and 1.4, it results that

$$\Theta(XA)=\begin{pmatrix} \rho(xa\text{-}y^*b) & \text{-}\rho(x^*b+ya) \\ \rho((x^*b+ya)^*) & \rho((xa\text{-}y^*b)^*) \end{pmatrix}$$
$$=\begin{pmatrix} \rho(xa\text{-}y^*b) & \text{-}\rho(x^*b+ya) \\ \rho((x^*b+ya)^*) & \rho((xa\text{-}y^*b)^*) \end{pmatrix}$$
$$=\begin{pmatrix} \rho(xa\text{-}y^*b) & \text{-}\rho(x^*b+ya) \\ \rho(xb^*+y^*a^*) & \rho(x^*a^*\text{-}yb^*) \end{pmatrix}.$$

$$\Theta(A)\Theta(X)=\begin{pmatrix} \rho(a) & \text{-}\rho(b) \\ \rho(b^*) & \rho(a^*) \end{pmatrix}\begin{pmatrix} \rho(x) & \text{-}\rho(y) \\ \rho(y^*) & \rho(x^*) \end{pmatrix}$$
$$=\begin{pmatrix} \rho(a)\rho(x)\text{-}\rho(b)\rho(y^*) & \text{-}\rho(a)\rho(y)\text{-}\rho(b)\rho(x^*) \\ \rho(b^*)\rho(x)+\rho(a^*)\rho(y^*) & \text{-}\rho(b^*)\rho(y)+\rho(a^*)\rho(x^*) \end{pmatrix}$$
$$=\begin{pmatrix} \rho(xa-y^*b) & -\rho(x^*b+ya) \\ \rho(xb^*+y^*a^*) & \rho(x^*a^*-yb^*) \end{pmatrix}.$$ □

Proposition 2.9. *Let $X\in\mathbb{H}_C$, $X=x+iy$, $x,y\in\mathbb{H}$, then:*

i) $\overrightarrow{X}=\begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix}\Theta(X)\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, *where* $1=I_4\in\mathcal{M}_4(\mathbb{R})$ *is the identity matrix,* $0=O_4\in\mathcal{M}_4(\mathbb{R})$ *is the zero matrix and* $\alpha\in\mathcal{M}_4(\mathbb{R})$ *as in Proposition* 2.5 iii).

ii) $\overrightarrow{XA}=\begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix}\Theta(A)\begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix}\overrightarrow{X}$.

iii) $\Gamma(A)\begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix}\Theta(B)\begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix}=\begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix}\Theta(B)\begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix}\Gamma(A)$ *for all* $A,B\in\mathbb{H}_C$.

Proof. i) We have $\begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix}\Theta(X)\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$=\begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix}\begin{pmatrix} \rho(x) & -\rho(y) \\ \rho(y^*) & \rho(x^*) \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$
$$=\begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix}\begin{pmatrix} \rho(x) \\ \rho(y^*) \end{pmatrix}=\begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix}\begin{pmatrix} \overrightarrow{x} \\ \overrightarrow{y^*} \end{pmatrix}$$
$$=\begin{pmatrix} \overrightarrow{x} \\ \alpha\overrightarrow{y^*} \end{pmatrix}=\begin{pmatrix} \overrightarrow{x} \\ \overrightarrow{y} \end{pmatrix}.$$

ii) $\overrightarrow{XA}=\begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix}\Theta(XA)\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$=\begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix}\Theta(A)\Theta(X)\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$
$$=\begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix}\Theta(A)\begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix}\Theta(X)\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$
$$=\begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix}\Theta(A)\begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix}\overrightarrow{X}.$$

iii) We obtain

$$\overrightarrow{AXB}=\overrightarrow{A(XB)}=\Gamma(A)\overrightarrow{XB}=\Gamma(A)\begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix}\Theta(B)\begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix}\overrightarrow{X}.$$

Since $\overrightarrow{AXB} = \overrightarrow{A(XB)} = \overrightarrow{(AX)B}$, it results that

$$\overrightarrow{(AX)B} = \begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix} \Theta(B) \begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix} \overrightarrow{AX}$$
$$= \begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix} \Theta(B) \begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix} \Gamma(A) \overrightarrow{X},$$

therefore we obtain the asked relation. □

Theorem 2.10. *With the above notations, the following relation is true:*

$$\Gamma^t(X) = M_1 \Theta(X) M_2,$$

where

$$M_1 = \begin{pmatrix} -A_1 & 0 \\ 0 & A_1 \end{pmatrix} \in \mathcal{M}_8(\mathbb{R}),$$

$$M_2 = \begin{pmatrix} -A_2 & 0 \\ 0 & A_2 \end{pmatrix} \in \mathcal{M}_8(\mathbb{R}) \text{ and}$$

$$A_1 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \in \mathcal{M}_4(\mathbb{R}),$$
$$A_2 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \in \mathcal{M}_4(\mathbb{R}).$$

Proof. First, we remark that $A_1 \rho(a) A_2 = \lambda^t(a)$. Indeed,

$$\begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} a_0 & -a_1 & -a_2 & -a_3 \\ a_1 & a_0 & a_3 & -a_2 \\ a_2 & -a_3 & a_0 & a_1 \\ a_3 & a_2 & -a_1 & a_0 \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} -a_1 & -a_0 & -a_3 & a_2 \\ -a_0 & a_1 & a_2 & a_3 \\ a_3 & a_2 & -a_1 & a_0 \\ -a_2 & a_3 & -a_0 & -a_1 \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} a_0 & a_1 & a_2 & a_3 \\ -a_1 & a_0 & a_3 & -a_2 \\ -a_2 & -a_3 & a_0 & a_1 \\ -a_3 & a_2 & -a_1 & a_0 \end{pmatrix} = \lambda^t(a).$$

We have

$$M_1 \Theta\left(\overline{X}\right) M_2$$
$$= \begin{pmatrix} -A_1 & 0 \\ 0 & A_1 \end{pmatrix} \begin{pmatrix} \rho(x) & -\rho(y) \\ \rho(y^*) & \rho(x^*) \end{pmatrix} \begin{pmatrix} -A_2 & 0 \\ 0 & A_2 \end{pmatrix}$$
$$= \begin{pmatrix} -A_1\rho(x) & A_1\rho(y) \\ A_1\rho(y^*) & A_1\rho(x^*) \end{pmatrix} \begin{pmatrix} -A_2 & 0 \\ 0 & A_2 \end{pmatrix}$$

$$=\begin{pmatrix} A_1\rho(x)A_2 & A_1\rho(y)A_2 \\ -A_1\rho(y^*)A_2 & A_1\rho(x^*)A_2 \end{pmatrix}=\begin{pmatrix} \lambda(x) & \lambda(y) \\ -\lambda(y^*) & \lambda(x^*) \end{pmatrix}$$

$$=\begin{pmatrix} \lambda(x) & -\lambda(y^*) \\ \lambda(y) & \lambda(x^*) \end{pmatrix}^t = \Gamma^t(x). \qquad \square$$

Remark 2.11. From Theorem 2.7 and Theorem 2.10, it results that

$$Q = -\frac{1}{4}N_1\Theta^t(X^*)N_2,$$

where $Q \in \mathbb{H}_C$ is a complex quaternion, $N_1 = M_8^t M_2^t$ and $N_2 = M_1^t M_8$.

Proposition 2.12. *For $Q \in \mathbb{H}_C$, $Q = a + ib$, we have:*

$$\det\Gamma(Q) = \det\Theta(Q) = n(aa^* + b^*b)^2 = n(a^*a + b^*b)^2.$$

Proof. We obtain:

$$\begin{aligned}\det\Gamma(Q) &= \det\begin{pmatrix} \lambda(a) & -\lambda(b^*) \\ \lambda(b) & \lambda(a^*) \end{pmatrix} \\ &= \det(\lambda(a)\lambda(a^*)+\lambda(b^*)\lambda(b)) \\ &= \det(\lambda(aa^*+b^*b)) = n(aa^*+b^*b)^2.\end{aligned}$$

For the second, we have:

$$\begin{aligned}\det\Theta(Q) &= \det\begin{pmatrix} \rho(a) & -\rho(b) \\ \rho(b^*) & \rho(a^*) \end{pmatrix} \\ &= \det(\rho(a)\rho(a^*)+\rho(b)\rho(b^*)) \\ &= \det(\rho(a^*a+b^*b)) = n(a^*a+b^*b)^2.\end{aligned}$$

By straightforward calculation, it results that $n(aa^* + b^*b)^2 = n(a^*a + b^*b)^2$. $\square$

3. Examples

The following sequence of numbers

$$0, 1, 1, 2, 3, 5, 8, 13, 21, \ldots,$$

with the nth term given by the formula:

$$f_n = f_{n-1} + f_{n-2},\ n \geq 2,$$

where $f_0 = 0, f_1 = 1$, is called the *Fibonacci numbers*.

In [3], the author defined and studied Fibonacci quaternions given by the formula:

$$F_n = f_n \cdot 1 + f_{n+1}e_2 + f_{n+2}e_3 + f_{n+3}e_4,$$

where f_n is the Fibonacci numbers,

$$e_m^2 = -1,\ \ m \in \{2, 3, 4\}$$

and

$$e_m e_q = -e_q e_m = \beta_{mq} e_t,\ \beta_{mq} \in \{-1, 1\}, m \neq q, m, q \in \{\, 2, 3, 4\},$$

β_{mq} and e_t being uniquely determined by e_m and e_q. F_n is called the nth Fibonacci quaternion. In the same paper, the author gave some relations for the nth Fibonacci quaternions, as for example the norm formula:

$$\mathbf{n}(F_n) = F_n\overline{F}_n = 3f_{2n+3},$$

where $\overline{F}_n = f_n \cdot 1 - f_{n+1}e_2 - f_{n+2}e_3 - f_{n+3}e_4$ is the conjugate of the F_n.

In the same paper, Horadam defined the nth complex Fibonacci numbers as follows:

$$q_n = f_n + if_{n+1}, i^2 = -1,$$

where f_n is the nth Fibonacci number.

Similarly, the nth complex Fibonacci quaternion is the element

$$Q_n = F_n + iF_{n+1}, i^2 = -1,$$

where F_n is the nth Fibonacci quaternion.

Example 3.1. For the real Fibonacci quaternion F_n, we have

$$\det(\lambda(F_n)) = \det(\rho(F_n)) = (\mathbf{n}(F_n))^2 = 9f_{2n+3}^2.$$

Example 3.2. The left matrix representation for a complex Fibonacci quaternion is the matrix:

$$\Gamma(Q_n) = \begin{pmatrix} f_n & -f_{n+1} & -f_{n+2} & -f_{n+3} & -f_{n+1} & f_{n+2} & -f_{n+3} & -f_{n+4} \\ f_{n+1} & f_n & -f_{n+3} & f_{n+2} & -f_{n+2} & -f_{n+1} & -f_{n+4} & f_{n+3} \\ f_{n+2} & f_{n+3} & f_n & -f_{n+1} & f_{n+3} & f_{n+4} & -f_{n+1} & f_{n+2} \\ f_{n+3} & -f_{n+2} & f_{n+1} & f_n & f_{n+4} & -f_{n+3} & -f_{n+2} & -f_{n+1} \\ f_{n+1} & -f_{n+2} & -f_{n+3} & -f_{n+4} & f_n & -f_{n+1} & f_{n+2} & f_{n+3} \\ f_{n+2} & f_{n+1} & -f_{n+4} & f_{n+3} & f_{n+1} & f_n & f_{n+3} & -f_{n+2} \\ f_{n+3} & f_{n+4} & f_{n+1} & -f_{n+2} & -f_{n+2} & -f_{n+3} & f_n & -f_{n+1} \\ f_{n+4} & -f_{n+3} & f_{n+2} & f_{n+1} & -f_{n+3} & f_{n+2} & f_{n+1} & f_n \end{pmatrix}.$$

By straightforward calculation, the determinant of the matrix $\Gamma(Q_n)$ is

$$\begin{aligned} \det\Gamma(Q_n) &= \left(f_n^2+2f_nf_{n+2}+2f_{n+2}^2+f_{n+4}^2+2f_{n+2}f_{n+4}\right)^2 \\ &\quad \cdot \left(f_n^2-2f_nf_{n+2}+4f_{n+1}^2+2f_{n+2}^2+4f_{n+3}^2+f_{n+4}^2-2f_{n+2}f_{n+4}\right)^2 \\ &= \left((f_n+f_{n+2})^2+(f_{n+2}+f_{n+4})^2\right)^2 \\ &\quad \cdot \left((f_{n+2}-f_n)^2+(f_{n+4}-f_{n+2})^2+4f_{n+1}^2+4f_{n+3}^2\right)^2 \\ &= \left((f_n+f_{n+2})^2+(f_{n+2}+f_{n+4})^2\right)^2 \left(5f_{n+1}^2+5f_{n+3}^2\right)^2 \\ &= 25\left((f_n+f_{n+2})^2+(f_{n+2}+f_{n+4})^2\right)^2 \left(f_{n+1}^2+f_{n+3}^2\right)^2. \end{aligned}$$

Example 3.3. The right matrix representation for a complex Fibonacci quaternion is the matrix:

$$\Theta(Q_n) = \begin{pmatrix} f_n & -f_{n+1} & -f_{n+2} & -f_{n+3} & -f_{n+1} & f_{n+2} & f_{n+3} & f_{n+4} \\ f_{n+1} & f_n & f_{n+3} & -f_{n+2} & -f_{n+2} & -f_{n+1} & -f_{n+4} & f_{n+3} \\ f_{n+2} & -f_{n+3} & f_n & f_{n+1} & -f_{n+3} & f_{n+4} & -f_{n+1} & -f_{n+2} \\ f_{n+3} & f_{n+2} & -f_{n+1} & f_n & -f_{n+4} & -f_{n+3} & f_{n+2} & -f_{n+1} \\ f_{n+1} & -f_{n+2} & f_{n+3} & f_{n+4} & f_n & -f_{n+1} & f_{n+2} & f_{n+3} \\ f_{n+2} & f_{n+1} & -f_{n+4} & f_{n+3} & f_{n+1} & f_n & -f_{n+3} & f_{n+2} \\ -f_{n+3} & f_{n+4} & f_{n+1} & f_{n+2} & -f_{n+2} & f_{n+3} & f_n & f_{n+1} \\ -f_{n+4} & -f_{n+3} & -f_{n+2} & f_{n+1} & -f_{n+3} & -f_{n+2} & -f_{n+1} & f_n \end{pmatrix}.$$

We have

$$\begin{aligned} \det\Gamma(Q_n) &= \left(f_n^2 + 2f_nf_{n+2} + 2f_{n+2}^2 + f_{n+4}^2 + 2f_{n+2}f_{n+4}\right)^2 \\ &\quad \cdot \left(f_n^2 - 2f_nf_{n+2} + 4f_{n+1}^2 + 2f_{n+2}^2 + 4f_{n+3}^2 + f_{n+4}^2 - 2f_{n+2}f_{n+4}\right)^2 \\ &= 25\left((f_n + f_{n+2})^2 + (f_{n+2} + f_{n+4})^2\right)^2 \left(f_{n+1}^2 + f_{n+3}^2\right)^2. \end{aligned}$$

Remark 3.4. A matrix representation for the complex Fibonacci quaternion was introduced in [2]. This matrix representation, denoted in the following with ε, is a pseudo-representation since $\varepsilon(XA) \neq \varepsilon(X)\varepsilon(A)$ or $\varepsilon(XA) \neq \varepsilon(A)\varepsilon(X)$, where $X, A \in \mathbb{H}_C$, $X = x + iy$, $A = a + ib$. Indeed, using the above notations, we can write the representation from [2] under the form

$$\varepsilon(A) = \begin{pmatrix} \rho^t(a) & \rho^t(b) \\ -\rho^t(b) & \rho^t(a) \end{pmatrix}.$$

By straightforward calculation, we have

$$\varepsilon(XA) = \begin{pmatrix} \rho^t(xa - y^*b) & \rho^t(x^*b + ya) \\ -\rho^t(x^*b + ya) & \rho^t(xa - y^*b) \end{pmatrix},$$

$$\varepsilon(X)\varepsilon(A) = \begin{pmatrix} \rho^t(xa - yb) & \rho^t(xb + ya) \\ -\rho^t(xb + ya) & \rho^t(xa - yb) \end{pmatrix},$$

and

$$\varepsilon(A)\varepsilon(X) = \begin{pmatrix} \rho^t(ax - by) & \rho^t(bx + ay) \\ -\rho^t(bx + ay) & \rho^t(ax - by) \end{pmatrix}.$$

From Fundamental Theorem of Algebra, it is known that any polynomial of degree n with coefficients in a field K has at most n roots in K. If the coefficients are in $\mathbb{H}$ (the division real quaternion algebra), the situation is different. For $\mathbb{H}$ over the real field, there it is a kind of a fundamental theorem of algebra: *If a polynomial has only one term of the greatest degree in* $\mathbb{H}$*then it has at least one root in* $\mathbb{H}$. (see [1] and [4]).

In the following, we will give two examples of complex quaternion equations with more than one greatest term with a unique solution or without solutions.

Example 3.5. Let $Q_n = F_n + iF_{n+1}$ be a complex Fibonacci quaternion and A a complex quaternion. We consider equations:

$$Q_nX - XQ_n = A \tag{3.1}$$

and

$$Q_nX + XQ_n = A. \tag{3.2}$$

If the equation (3.1) has a solution, then this solution is not unique, but the equation (3.2) has a unique solution. Indeed, using the vector representation, Proposition 2.5 and Proposition 2.9, equation (3.1) becomes:

$$\left(\Gamma\left(Q_n\right)-\begin{pmatrix}1&0\\0&\alpha\end{pmatrix}\Theta\left(Q_n\right)\begin{pmatrix}1&0\\0&\alpha\end{pmatrix}\right)\overrightarrow{X}=\overrightarrow{A}.$$

We obtain that the matrix

$$B=\Gamma\left(Q_n\right)-\begin{pmatrix}1&0\\0&\alpha\end{pmatrix}\Theta\left(Q_n\right)\begin{pmatrix}1&0\\0&\alpha\end{pmatrix}$$
$$=\begin{pmatrix}0&0&0&0&0&0&0&0\\0&0&-2f_{n+3}&2f_{n+2}&0&0&-2f_{n+4}&2f_{n+3}\\0&2f_{n+3}&0&-2f_{n+1}&2f_{n+3}&0&-2f_{n+1}&0\\0&-2f_{n+2}&2f_{n+1}&0&2f_{n+4}&0&0&-2f_{n+1}\\0&0&-2f_{n+3}&-2f_{n+4}&0&0&2f_{n+2}&2f_{n+3}\\0&0&0&0&0&0&0&0\\0&2f_{n+4}&2f_{n+1}&0&-2f_{n+2}&0&0&-2f_{n+1}\\0&-2f_{n+3}&0&2f_{n+1}&-2f_{n+3}&0&2f_{n+1}&0\end{pmatrix}$$

has $\det B=0$ and $rankB=4$, as we can find by straightforward calculation. Therefore, if the equation (3.1) has a solution, this solution is not unique.

In the same way, the equation (3.2) becomes

$$\left(\Gamma\left(Q_n\right)+\begin{pmatrix}1&0\\0&\alpha\end{pmatrix}\Theta\left(Q_n\right)\begin{pmatrix}1&0\\0&\alpha\end{pmatrix}\right)\overrightarrow{X}=\overrightarrow{A}.$$

We obtain that the matrix

$$D=\Gamma\left(Q_n\right)+\begin{pmatrix}1&0\\0&\alpha\end{pmatrix}\Theta\left(Q_n\right)\begin{pmatrix}1&0\\0&\alpha\end{pmatrix}$$
$$=\begin{pmatrix}2f_n&-2f_{n+1}&-2f_{n+2}&-2f_{n+3}&-2f_{n+1}&2f_{n+2}&-2f_{n+3}&-2f_{n+4}\\2f_{n+1}&2f_n&0&0&-2f_{n+2}&-2f_{n+1}&0&0\\2f_{n+2}&0&2f_n&0&0&2f_{n+4}&0&2f_{n+2}\\2f_{n+3}&0&0&2f_n&0&-2f_{n+3}&-2f_{n+2}&0\\2f_{n+1}&-2f_{n+2}&0&0&2f_n&-2f_{n+1}&0&0\\2f_{n+2}&2f_{n+1}&-2f_{n+4}&2f_{n+3}&2f_{n+1}&2f_n&2f_{n+3}&-2f_{n+2}\\2f_{n+3}&0&0&-2f_{n+2}&0&-2f_{n+3}&2f_n&0\\2f_{n+4}&0&2f_{n+2}&0&0&2f_{n+2}&0&2f_n\end{pmatrix}$$

has

$$\begin{aligned}&\det D\\&=256\left(f_n-f_{n+2}\right)^2\left(f_n+f_{n+2}\right)^2\left(f_n^2+2f_nf_{n+2}+2f_{n+2}^2+f_{n+4}^2+2f_{n+2}f_{n+4}\right)\\&\quad\cdot\left(f_n^2-2f_nf_{n+2}+4f_{n+1}^2+2f_{n+2}^2+4f_{n+3}^2+f_{n+4}^2-2f_{n+2}f_{n+4}\right)\\&=1280f_{n+1}^2\left(f_n+f_{n+2}\right)^2\left(\left(f_n+f_{n+2}\right)^2+\left(f_{n+2}+f_{n+4}\right)^2\right)\left(f_{n+1}^2+f_{n+3}^2\right).\end{aligned}$$

It results $\det D\neq 0$, therefore the equation (3.2) has a unique solution.

Example 3.6. With the above notations, the matrix

$$\delta\left(Q_n\right) = \Gamma\left(Q_n\right) - \Theta\left(Q_n\right)$$

is an invertible matrix.

Indeed,

$$\delta\left(Q_n\right) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -2f_{n+3} & -2f_{n+4} \\ 0 & 0 & -2f_{n+3} & 2f_{n+2} & 0 & 0 & 0 & 0 \\ 0 & 2f_{n+3} & 0 & -2f_{n+1} & 2f_{n+3} & 0 & 0 & 2f_{n+2} \\ 0 & -2f_{n+2} & 2f_{n+1} & 0 & 2f_{n+4} & 0 & -2f_{n+2} & 0 \\ 0 & 0 & -2f_{n+3} & -2f_{n+4} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2f_{n+3} & -2f_{n+2} \\ 2f_{n+3} & 0 & 0 & -2f_{n+2} & 0 & -2f_{n+3} & 0 & -2f_{n+1} \\ 2f_{n+4} & 0 & 2f_{n+2} & 0 & 0 & 2f_{n+2} & 2f_{n+1} & 0 \end{pmatrix}$$

and

$$\det \delta\left(Q_n\right) = 256\left(f_{n+3}\right)^4 \left(f_{n+2} + f_{n+4}\right)^4$$

is different from zero.

Conclusions

In this paper we introduced two real matrix representation for the complex quaternions and we investigated some of the properties of these representations. Because of their various applications to complex quaternions and to matrices of complex quaternions, this paper can be regarded as a starting point for a further research of these representations.

Acknowledgements. Authors thank referee for his/her suggestions which help us to improve this paper.

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Received: September 10, 2012.
Accepted: November 26, 2012.